

Mathematics Comps, Carleton College:
Topology of Second Order Tensor Fields
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In this paper, we will provide an overview of the topology of real symmetric second-order tensor fields, focusing on tensor fields in \mathbb{R}^2 and \mathbb{R}^3 . These tensor fields have many applications in physics, such as modeling physical stress in different directions at given points in space. Tensor fields are also used in diffusion MRI brain imaging, to identify axon fibers and other structures in the brain [1]. In section 1, we will define tensors and tensor field topologies, providing the foundation for an in-depth study of symmetric second-order tensor field topologies in \mathbb{R}^2 and \mathbb{R}^3 , in sections 2 and 3 respectively. The majority of material in sections 2 and 3 is based on chapters 13 and 14 of *Visualization and Processing of Tensor Fields* edited by Weickert and Hagen [6, 7]. In general, we will focus on the characterization and computation of degenerate points in real symmetric second-order tensor fields. Finally, we conclude in section 4 with a brief explanation of tensor field applications to diffusion tensor imaging in neuroscience, justifying further study of second order tensor field topology.

1 Tensor Field Topology

In this section, we define tensors, tensor fields, and the topology of a second order tensor field. This paper will focus on real symmetric second order smooth tensor fields, which can be understood as smooth assignments of symmetric linear maps $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to each point in \mathbb{R}^n . Section 1.1 will cover the linear algebra background necessary to define tensors in section 1.2. Then, section 1.3 will justify the representation of type $(1, 1)$ tensors as linear maps, and section 1.4 will justify the representation of real symmetric second order tensors as symmetric linear maps. Finally, section 1.5 will define tensor fields and the topology of a second order tensor field.

1.1 Background: Linear Algebra

Before defining a tensor, we briefly review the concepts of vector spaces, linear maps, eigenvectors, and dual spaces. For the most part, this section will be a collection of definitions, which will be referred to in later sections.

A vector space V over a field \mathbb{F} is a set of elements with a properly defined addition operation ($V \times V \rightarrow V$) and scalar multiplication operation ($\mathbb{F} \times V \rightarrow V$). The precise requirements of these two operations can be found in a standard linear algebra textbook such as [5]. Elements of V are called vectors, and a set $\{e_1, \dots, e_n\} \subseteq V$ is a basis of V if and only if every $v \in V$ can be written as a unique linear combination:

$$v = v_1 e_1 + \dots + v_n e_n \quad \text{where } v_1, \dots, v_n \in \mathbb{F}$$

This representation of v leads to the common vector notation: $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

Every basis of V has n elements, and we say that V has dimension n . However, it is important to remember that the values v_1, \dots, v_n depend on the choice of basis for V .

Next, given vector spaces U and V with corresponding dimensions m and n , a linear map is defined as a function $L : U \rightarrow V$ satisfying $L(\alpha u_0 + \beta u_1) = \alpha L(u_0) + \beta L(u_1)$ for all $u_0, u_1 \in U$ and all $\alpha, \beta \in \mathbb{F}$. Given bases for U and V , any linear map can be expressed as a unique $n \times m$ matrix with values in \mathbb{F} . The construction of this matrix representation can be found in linear algebra textbooks. Extending upon linear maps, a multilinear map $M : V_1 \times \dots \times V_k \rightarrow W$ is a map that is linear separately in each variable. More formally, for each v_i , if all other v_j are held constant, then $M(v_1, \dots, v_k)$ is a linear map of v_i . The concept of multilinear maps will be important when defining tensors.

Next, given a linear map $L : V \rightarrow V$, an eigenvector of L is a vector $v \in V$ satisfying $L(v) = \lambda v$ for some $\lambda \in \mathbb{F}$. Then, λ is called the eigenvalue corresponding to the eigenvector v . Note that any scalar multiple of v is also an eigenvector of L , and multiple eigenvectors can correspond to the same eigenvalue λ .

Lastly, we define the dual space of an n -dimensional vector space V . First, a linear form on V is a linear map $\omega : V \rightarrow \mathbb{F}$. Linear forms are also called dual vectors or covectors. The dual space of V is denoted V^* , defined as the set of all linear forms on V . It can be shown that V^* is an n -dimensional vector space over \mathbb{F} , where addition and scalar multiplication are defined by:

$$(\omega_0 + \omega_1)(v) = \omega_0(v) + \omega_1(v)$$

$$(\alpha\omega)(v) = \alpha\omega(v)$$

In fact, when $V = \mathbb{R}^n$ (or any other Hilbert space¹), then V is isomorphic to its dual space V^* [2]. For any vector space V , the dual of the dual space V^{**} is isomorphic to V . Finally, given any vector space V with basis $\{e_1, \dots, e_n\}$, there exists a corresponding dual basis $\{\epsilon_1, \dots, \epsilon_n\}$ such that $\epsilon_i(e_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

1.2 Tensors

1.2.1 Definition and Components of a Tensor

Given an n -dimensional vector space V over a field \mathbb{F} , an (r, s) -tensor is a multilinear map [4]:

$$T : \underbrace{V \times \dots \times V}_r \times \underbrace{V^* \times \dots \times V^*}_s \rightarrow \mathbb{F} \quad (1)$$

Then, we say that T has type (r, s) and order (or rank) $r + s$. We will show that given a basis of V , T can be represented by n^{r+s} values in \mathbb{F} , called components of T . This gives rise to the representation of T as an extended matrix with $r + s$ indices (i.e. an $\underbrace{n \times \dots \times n}_{r+s}$ matrix).

Suppose $\{E_1, \dots, E_n\}$ is a basis of V and $\{E^1, \dots, E^n\}$ is the corresponding dual basis of V^* . Here, we use capital letters with subscripts to represent vectors, and we use capital letters with superscripts to represent elements of the dual space; superscripts here do not indicate exponentiation. From now on, we will also use Einstein notation to represent sums; if an index appears more than once in an expression, then the expression is summed over all possible values for that index. For instance, a vector $A_1 \in V$ can be written as the linear combination:

$$A_1 = a_1^{i_1} E_{i_1} = \sum_{i_1=1}^n a_1^{i_1} E_{i_1}$$

Note again that superscripts for a_1 do not indicate exponentiation, but rather serve only as indices.

Now, we derive the n^{r+s} components of T . By representing each input vector or covector of T with respect to our defined bases, we have:

$$T(A_1, \dots, A_r, B^1, \dots, B^s) = T(a_1^{i_1} E_{i_1}, \dots, a_r^{i_r} E_{i_r}, b_{j_1}^1 E^{j_1}, \dots, b_{j_s}^s E^{j_s})$$

Because T is multilinear, we can expand this expression into the sum of n^{r+s} terms, one for each possible choice of $i_1, \dots, i_r, j_1, \dots, j_s$. Intuitively, we are repeatedly expanding with respect to each subsequent input vector or covector; each expansion converts each term in the existing expansion into n new terms. Written in Einstein notation, we have the expansion:

$$T(A_1, \dots, A_r, B^1, \dots, B^s) = T(E_{i_1}, \dots, E_{i_r}, E^{j_1}, \dots, E^{j_s}) a_1^{i_1} \dots a_r^{i_r} b_{j_1}^1 \dots b_{j_s}^s$$

¹A Hilbert space H is a vector space with a positive symmetric inner product $\langle \cdot, \cdot \rangle$ such that the norm $|v| = \sqrt{\langle v, v \rangle}$ makes H a complete metric space. A complete metric space is a metric space in which every Cauchy sequence is convergent.

Note that all a and b values are determined entirely by the input vectors and covectors once the bases have been defined. Thus, the tensor T is defined entirely by its value for each $T(E_{i_1}, \dots, E_{i_r}, E^{j_1}, \dots, E^{j_s})$. We denote this value $t_{i_1, \dots, i_r}^{j_1, \dots, j_s}$ for each choice of $i_1, \dots, i_r, j_1, \dots, j_s$. Essentially, this is the value of T when one basis vector is chosen for each input parameter. There are n^{r+s} distinct ways to choose one basis vector for each input parameter, and thus T can be represented by the n^{r+s} different components $t_{i_1, \dots, i_r}^{j_1, \dots, j_s}$ with respect to the given bases.

1.2.2 Examples of Tensors

Now that we have defined tensors, we will briefly provide examples of mathematical concepts expressible as tensors, motivating further study of tensors. As a basic example, scalars in \mathbb{F} are type $(0,0)$ tensors. Covectors in the dual space V^* are type $(1,0)$ tensors, mapping vectors to elements of \mathbb{F} . In contrast, vectors in V correspond to type $(0,1)$ tensors (mapping covectors to elements of \mathbb{F}), functioning as elements of V^{**} , which is isomorphic to V .

Furthermore, bilinear forms such as inner products can be expressed as type $(2,0)$ tensors. The determinant of an $n \times n$ matrix is an example of a type $(n,0)$ tensor [4]. Finally, linear maps can be expressed as type $(1,1)$ tensors; these type $(1,1)$ tensors will be the focus of the remainder of this paper. In the next section, we will prove the correspondence between type $(1,1)$ tensors and linear maps.

1.3 Second Order Tensors as Linear Maps

This section will show that we can consider type $(1,1)$ tensors as linear maps from a vector space to itself. Recall that a type $(1,1)$ tensor is a multilinear map $T : V \times V^* \rightarrow \mathbb{F}$. As shown in the previous section, given a basis for V , type $(1,1)$ tensors can be represented as $n \times n$ matrices; these matrices are also used to represent linear maps $L : V \rightarrow V$. In this section, we will provide an isomorphism between type $(1,1)$ tensors and linear maps.

First, let \mathcal{T}_1^1 be the set of all type $(1,1)$ tensors over the vector space V , and let $\mathcal{L}(V, V)$ be the set of all linear maps from V to itself. We define the function $\Phi : \mathcal{L}(V, V) \rightarrow \mathcal{T}_1^1$ by:

$$\Phi : L \mapsto \left(T : (A, B) \mapsto B(L(A)) \right) \quad (2)$$

Intuitively, T applies L to the vector A and then applies the covector B , resulting in an element of \mathbb{F} as desired. We claim that Φ is an isomorphism.

First, we show that Φ is bijective. To show injectivity, consider some $L_1, L_2 \in \mathcal{L}(V, V)$ such that $\Phi(L_1) = \Phi(L_2)$. Then, for all $A \in V$ and $B \in V^*$ we have:

$$\begin{aligned} (\Phi(L_1))(A, B) &= (\Phi(L_2))(A, B) \\ B(L_1(A)) &= B(L_2(A)) \end{aligned}$$

Because this equation holds for all linear maps $B : V \rightarrow \mathbb{F}$, we must have $L_1(A) = L_2(A)$. Then, because the equation holds for all $A \in V$, we must have $L_1 = L_2$. Thus Φ is injective.

To show surjectivity, consider some $T \in \mathcal{T}_1^1$. Let $\{E_1, \dots, E_n\}$ be a basis of V and $\{E^1, \dots, E^n\}$ the corresponding dual basis. Note that we will be using the Einstein summation convention introduced in the previous section. Then:

$$\begin{aligned} T(A, B) &= T(a^j E_j, b_i E^i) \\ &= T(E_j, E^i) a^j b_i \end{aligned}$$

Consider the linear map L with the matrix representation M (with respect to the given bases) defined by $M_{ij} = T(E_j, E^i)$. Writing out the matrix M , we have:

$$\begin{pmatrix} T(E_1, E^1) & \cdots & T(E_n, E^1) \\ \vdots & \ddots & \vdots \\ T(E_1, E^n) & \cdots & T(E_n, E^n) \end{pmatrix}$$

Note that by the rules of matrix multiplication, we have for all $A \in V$:

$$\begin{aligned} L(A) &= L(a^j E_j) \\ &= M_{ij} a^j E_i \\ &= T(E_j, E^i) a^j E_i \end{aligned}$$

Then, for all $A \in V$ and $B \in V^*$ we have:

$$\begin{aligned} (\Phi(L))(A, B) &= B(L(A)) \\ &= b_k E^k (T(E_j, E^i) a^j E_i) \\ &= a^j b_k T(E_j, E^i) E^k (E_i) \end{aligned}$$

Because the dual basis ensures that $E^k(E_i) = \delta_{ik}$, we need only consider terms where $i = k$:

$$(\Phi(L))(A, B) = a^j b_i T(E_j, E^i)$$

Note that this equation holds for any choice of bases for V and V^* . Thus we have shown that $(\Phi(L))(A, B) = T(A, B)$ for all $A \in V, B \in V^*$, so $\Phi(L) = T$. Therefore Φ is surjective.

Finally, we show that Φ is a homomorphism (i.e. Φ is linear). We see:

$$\begin{aligned} \Phi(\alpha L_1 + \beta L_2) &= \left(T : (A, B) \mapsto B((\alpha L_1 + \beta L_2)(A)) \right) \\ &= \left(T : (A, B) \mapsto \alpha B(L_1(A)) + \beta B(L_2(A)) \right) \\ &= \alpha \Phi(L_1) + \beta \Phi(L_2) \end{aligned}$$

Thus Φ is linear. Because Φ is a bijective homomorphism, Φ is an isomorphism.

The isomorphism Φ provides an explanation for the representation of type $(1, 1)$ tensors as linear maps. Note that given a type $(1, 1)$ tensor T , $\Phi^{-1}(T)$ is the linear map L such that $T(A, B) = B(L(A))$ for all $A \in V, B \in V^*$. This paper will focus on type $(1, 1)$ tensors, representing these tensors as linear maps.

1.4 Real Symmetric Second Order Tensors

The remainder of this paper will focus on second order tensors in the vector space $V = \mathbb{R}^n$. Because we will be using $V = \mathbb{R}^n$, we have $V \cong V^*$, so type $(1, 1)$ tensors are theoretically equivalent to type $(0, 2)$ and type $(2, 0)$ tensors. Thus, this paper will consider real second order tensors in general, represented as linear maps.

Furthermore, we will restrict our study to symmetric second order tensors. A symmetric tensor is a tensor T that is invariant under any permutation of its vector arguments. Formally, for every permutation $\sigma \in S_r$:

$$T(A_1, \dots, A_r) = T(A_{\sigma(1)}, \dots, A_{\sigma(r)}) \quad (3)$$

Given any basis for V , it can be shown that symmetric tensors must have components satisfying $t_{i_1, \dots, i_r} = t_{i_{\sigma(1)}, \dots, i_{\sigma(r)}}$ for all $\sigma \in S_r$. In other words, the indices i can be freely permuted. Thus, in the case of a type $(2, 0)$ symmetric tensor, the n^2 components must satisfy $t_{i_1, i_2} = t_{i_2, i_1}$ for all $0 < i_1, i_2 \leq n$.

In the next section, we will show that type $(2, 0)$ real symmetric tensors can be represented as symmetric linear maps. A symmetric linear map $L : V \rightarrow V$ satisfies $\langle L(A_1), A_2 \rangle = \langle A_1, L(A_2) \rangle$ for all $A_1, A_2 \in V$, where $\langle \cdot, \cdot \rangle$ is an inner product operator. Given any orthonormal basis of V , the matrix representation M of L is a symmetric matrix (i.e. $M^T = M$), which greatly simplifies the study of real symmetric second order tensors. Notably, real symmetric matrices have orthogonal eigenvectors, and all eigenvalues of a real symmetric matrix are real.

1.4.1 Symmetric Second Order Tensors and Symmetric Linear Maps

We now show that type $(2, 0)$ real symmetric tensors can be represented as symmetric linear maps. In fact, this proof holds for type $(2, 0)$ symmetric tensors in any Hilbert space.

First, the Riesz representation theorem [2] states that if V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then there exists an isomorphism $\Psi : V \rightarrow V^*$ such that for all $A_1, A_2 \in V$:

$$\Psi(A_1)(A_2) = \langle A_1, A_2 \rangle \quad (4)$$

Then, we define a function Θ mapping type $(2, 0)$ tensors to type $(1, 1)$ tensors:

$$\Theta : T_2^0 \mapsto \left(T_1^1 : (A, B) \mapsto T_2^0(A, \Psi^{-1}(B)) \right) \quad (5)$$

Here, T_1^1 applies T_2^0 to the vectors A and $\Psi^{-1}(B)$. First, note that T_1^1 is a well-defined tensor (i.e. T_1^1 is multilinear) because Ψ^{-1} is linear. Furthermore, Θ is bijective because Ψ^{-1} is bijective. We also have that Θ itself is linear:

$$\begin{aligned} \Theta(\alpha T_2^0 + \beta T_2^{0'}) &= \left(T_1^1 : (A, B) \mapsto (\alpha T_2^0 + \beta T_2^{0'})(A, \Psi^{-1}(B)) \right) \\ &= \left(T_1^1 : (A, B) \mapsto \alpha T_2^0(A, \Psi^{-1}(B)) + \beta T_2^{0'}(A, \Psi^{-1}(B)) \right) \\ &= \alpha \Theta(T_2^0) + \beta \Theta(T_2^{0'}) \end{aligned}$$

Thus $\Theta : \mathcal{T}_2^0 \rightarrow \mathcal{T}_1^1$ is an isomorphism.

Then, we can define an isomorphism from type $(2, 0)$ tensors to linear maps using the isomorphism $\Phi : \mathcal{L}(V, V) \rightarrow \mathcal{T}_1^1$ found in Section 1.3 (Equation 2). As the composition of two isomorphisms, $\Phi^{-1} \circ \Theta : \mathcal{T}_2^0 \rightarrow \mathcal{L}(V, V)$ is an isomorphism. Written out entirely, we have:

$$\Phi^{-1} \circ \Theta(T_2^0) = \left(L \text{ such that for all } A \in V, B \in V^* : T_2^0(A, \Psi^{-1}(B)) = B(L(A)) \right) \quad (6)$$

Finally, we show that $\Phi^{-1} \circ \Theta(T_2^0)$ is a symmetric linear map if and only if T_2^0 is a symmetric tensor.

$$\begin{aligned} T_2^0(A_1, A_2) &= T_2^0(A_2, A_1) && \forall A_1, A_2 \in V \\ \iff T_2^0(A, \Psi^{-1}(B)) &= T_2^0(\Psi^{-1}(B), A) && \forall A \in V, B \in V^* \\ \iff B(L(A)) &= \Psi(A)(L(\Psi^{-1}(B))) \\ &\text{(by the way we defined } T_2^0) \\ \iff \langle \Psi^{-1}(B), L(A) \rangle &= \langle A, L(\Psi^{-1}(B)) \rangle \\ &\text{(by the way we defined } \Psi) \\ \iff \langle \Psi^{-1}(B), L(A) \rangle &= \langle L(\Psi^{-1}(B)), A \rangle \\ &\text{(because } \langle \cdot, \cdot \rangle \text{ is symmetric)} \\ \iff \langle A_1, L(A_2) \rangle &= \langle L(A_1), A_2 \rangle && \forall A_1, A_2 \in V \end{aligned}$$

Thus, $\Phi^{-1} \circ \Theta$ is also an isomorphism from type $(2, 0)$ symmetric tensors to symmetric linear maps (assuming V is a Hilbert space). Therefore, real symmetric second order tensors can be represented by real symmetric $n \times n$ matrices given any orthonormal basis of V .

1.5 Tensor Fields and Tensor Field Topology

We are now able to define tensor fields and the topology of a second order tensor field. Given a topological space X , a tensor field $F_r^s : X \rightarrow \mathcal{T}_r^s$ assigns a type (r, s) tensor to each point in X . This paper will consider only smooth tensor fields, tensor fields in which each component of the tensor $F_r^s(p)$ varies smoothly with respect to p .

Furthermore, the remainder of this paper will consider only real symmetric second order tensor fields, which we will simply refer to as tensor fields. We will assume that $X = V = \mathbb{R}^n$. Because

real symmetric second order tensors can be represented as symmetric linear maps, we will generally consider tensor fields as assignments of symmetric linear maps $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to each point in \mathbb{R}^n .

Then, the topology of a tensor field F consists of the eigenvectors and corresponding eigenvalues of the linear map associated with each point $p \in \mathbb{R}^n$. Because real symmetric linear maps have all real eigenvalues and orthogonal eigenvectors, these eigenvectors and eigenvalues completely determine each linear map, and thus they completely determine the tensor field F . Thus, this definition for the topology of a tensor field is used only loosely; it is common to redefine the tensor field topology to consider only the relevant features of the entire topology. We will now turn to some of the common features used to define tensor field topologies.

1.5.1 Eigenvector Fields and Degenerate Points

The eigenvalues at each point $p \in \mathbb{R}^n$ can be ordered using the standard ordering of the real numbers. Then, the i^{th} eigenvalue is associated with the corresponding eigenvector, called the i^{th} eigenvector. Then, the i^{th} eigenvector field maps each point $p \in \mathbb{R}^n$ to the i^{th} eigenvector at p .

However, the i^{th} eigenvector at p may not be well-defined if the linear map at p has at least two equal eigenvalues (corresponding to at least two orthogonal eigenvectors). Thus, degenerate tensors are defined as tensors whose associated linear map has at least two equal eigenvalues; points associated with degenerate tensors in the tensor field are called degenerate points. An isotropic tensor is a tensor whose associated linear map has all equal eigenvalues; when $n = 2$, degenerate tensors are equivalent to isotropic tensors.

Then, we can equivalently define the topology of a tensor field as the set of n eigenvector fields with corresponding eigenvalue fields, along with the set of degenerate points with associated degenerate tensors. Because the tensor field is smooth, each eigenvector and eigenvalue field is smooth at all points where the eigenvalue λ_i is unique.

1.5.2 Tensor Field Lines

Lastly, a tensor field line in an eigenvector field is a curve that is tangent to the direction of the eigenvector field at all non-degenerate points. Because eigenvectors in eigenvector fields (in contrast with vectors in vector fields) have no magnitude and always point in two directions separated by 180° , the existence and uniqueness of tensor field lines cannot be guaranteed given an initial point p .

However, tensor field line integration can be formally defined using covering spaces and vector field streamlines. Conceptually, the eigenvector field is covered by two normalized vector fields with opposing directions, and vector field streamlines are projected onto tensor field lines using the path lifting property [6]. The detailed construction of these tensor field lines is beyond the scope of this paper, and we will generally assume that tensor field lines can be found in tensor fields.

Finally, we note that tensor field lines can only cross at degenerate points. At non-degenerate points, there is one unique tangent direction for any given eigenvector field. At degenerate points, there are at least two orthogonal eigenvectors corresponding to the same eigenvalue; thus, the eigenspace for the duplicate eigenvalue is at minimum two-dimensional, allowing for infinitely many tangent directions. This allows multiple distinct tensor field lines to pass through the same degenerate point. Along with degenerate points, tensor field lines are often used to characterize the topology of a tensor field.

2 Topology of Two-Dimensional Tensor Fields

We have defined the topology of a tensor field as the collection of eigenvector and eigenvalue fields, degenerate points, and degenerate tensors in the tensor field. This section will focus on two-dimensional tensor fields (i.e. $V = \mathbb{R}^2$), studying types of degenerate points and methods of simplifying the tensor field topology. Two-dimensional tensor fields have two associated eigenvector fields, corresponding to the eigenvalues $\lambda_1 \geq \lambda_2$ at each point $p \in \mathbb{R}^2$. We call these eigenvector fields the major and minor eigenvector fields respectively.

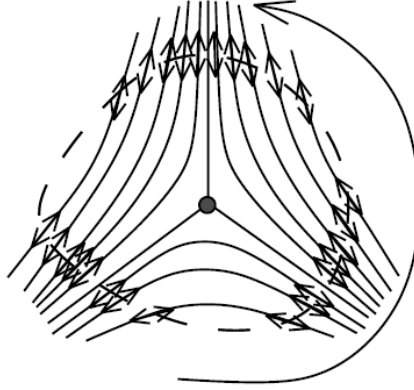


Figure 1: The tensor index of a path is the number of counterclockwise rotations made by the eigenvectors of the tensor field, when traveling counterclockwise along the path. The tensor index of the path above is $-\frac{1}{2}$. Lines in the figure represent tensor field lines, and double-sided arrows represent eigenvector directions. Figure taken from [6].

In section 2.1, we will introduce the concept of the tensor index of a degenerate point. This tensor index is helpful in characterizing sectors of a degenerate point, introduced in section 2.2. Next, the invariant δ in section 2.3 is used to characterize and classify linear degenerate points in section 2.4. Nonlinear degenerate points are addressed in section 2.5. Then, we consider how tensor field topologies change over time (section 2.6) and how tensor field topologies can be computed in practice (section 2.7). We address ways to simplify complex topologies (section 2.8) and ways to track topology changes over time (section 2.9). This provides a broad overview of methods for understanding the topology of two-dimensional tensor fields.

2.1 Tensor Index

First, we introduce the concept of a tensor index [3] in a 2D tensor field. The tensor index serves as an important feature of 2D tensor field topology, assigning half-integer values (e.g. $0, \frac{1}{2}, -\frac{1}{2}, 1$) to degenerate points in the tensor field.

In order to define the tensor index of a degenerate point, we first define the tensor index of a simple closed path in the tensor field space \mathbb{R}^2 . A simple closed path γ in \mathbb{R}^2 forms a continuous loop in \mathbb{R}^2 that does not intersect itself except for its endpoints.² If there are no degenerate points along the path γ , then the tensor index of γ in the tensor field F is defined as the number of counterclockwise rotations made by the eigenvectors of F when traveling once counterclockwise along γ (see Figure 1). Because eigenvectors rotate smoothly along γ and eigenvectors have two equivalent orientations (e.g. a half-rotation results in the original orientation because eigenvectors are bi-directional), the tensor index must be an integer multiple of $\frac{1}{2}$. Additionally, because the eigenvector fields of F are orthogonal to one another at all points along γ , the tensor index does not depend on the choice of eigenvector field from which to pull eigenvectors.

It can be shown that any path enclosing no degenerate points has tensor index zero. Furthermore, the tensor index of any path enclosing a single degenerate point p is equal to the tensor index of any other path enclosing only the same degenerate point [3]. We then say that p has tensor index $I_F(p)$ equal to the tensor index of any of these paths. This definition is applicable only to isolated degenerate points in the tensor field; an isolated degenerate point is a degenerate point p such that there exists an open set $M \subset \mathbb{R}^2$ containing p where M contains no other degenerate points. Non-isolated degenerate points cannot be enclosed by paths enclosing only one degenerate point.

Finally, it can be shown that the tensor index of any path enclosing a finite number of degenerate points is equal to the sum of the tensor indices of the enclosed degenerate points. Because

²Formally, a simple closed path in \mathbb{R}^2 is a continuous function $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ such that $\gamma(0) = \gamma(1)$ and γ is injective when restricted to the domain $[0, 1)$.

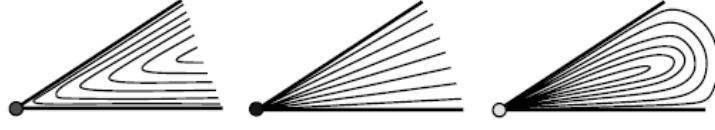


Figure 2: An example of a hyperbolic, parabolic, and elliptic sector. Figure taken from [6].

the tensor index relies only on the direction of eigenvectors along the path γ but still encapsulates information about the enclosed degenerate points, the tensor index is useful in understanding stable features of a tensor field topology. The next sections will further elaborate on the relationship between an isolated degenerate point and its tensor index; degenerate points in two-dimensional tensor fields are typically isolated [7]. For the remainder of section 2, isolated degenerate points will be referred to simply as degenerate points.

2.2 Sectors and Separatrices of a Degenerate Point

Given the major or minor eigenvector field around an isolated degenerate point p , the tensor field lines around p are characterized by a set of sectors, angular regions around p in which the tensor field lines exhibit a consistent behavior. With the exception of foci and centers (addressed in section 2.5.1), degenerate points have sectors that fall into three categories:

- Hyperbolic sectors, where each tensor field line deflects away from p .
- Parabolic sectors, where each tensor field line travels towards or away from p .
- Elliptic sectors, where each tensor field line begins and ends at p .

Figure 2 shows examples of the three types of sectors. For convenience, adjacent parabolic sectors are always merged into one parabolic sector. We define a separatrix as a tensor field line on the boundary of a hyperbolic sector; identifying boundaries between sectors will be addressed in section 2.4.1.

A degenerate point is then characterized within the major or minor eigenvector field by its sectors when traveling counterclockwise around the point. The foci and centers addressed in section 2.5.1 are characterized by their lack of any of the three types of sectors. Assuming a degenerate point p is not a center or focus, we can quantify the amount an eigenvector rotates when moving counterclockwise through each sector around p :

- Hyperbolic sectors with angle θ rotate the eigenvector counterclockwise by $\theta - \pi$.
- Parabolic sectors with angle θ rotate the eigenvector counterclockwise by θ .
- Elliptic sectors with angle θ rotate the eigenvector counterclockwise by $\theta + \pi$.

Suppose p has n_h hyperbolic sectors, n_b parabolic sectors, and n_e elliptic sectors with corresponding angles α_i , β_i , and γ_i . Then, the tensor index $I_T(p)$ satisfies:

$$2\pi I_T(p) = \sum_{i=0}^{n_h} (\alpha_i - \pi) + \sum_{i=0}^{n_b} \beta_i + \sum_{i=0}^{n_e} (\gamma_i + \pi)$$

Because the sum of the sector angles is equal to 2π , we have:

$$\begin{aligned} 2\pi I_T(p) &= 2\pi - n_h\pi + n_e\pi \\ I_T(p) &= 1 + \frac{n_e - n_h}{2} \end{aligned} \tag{7}$$

In fact, this equation holds for all degenerate points p , including focus and center points (which have tensor index 1 and $n_e = n_h = n_p = 0$). Because the tensor index $I_T(p)$ depends only on the tensor field F and the degenerate point p , we have that $n_e - n_h$ is invariant with respect to the eigenvector field chosen (major or minor). That said, different eigenvector fields at the same degenerate point may still have different numbers of hyperbolic, parabolic, and elliptic sectors.

2.3 Decomposing the Tensor Field and the Invariant δ

In this section, we will introduce ways of decomposing the tensor field around a degenerate point, leading to the invariant δ , which is used to distinguish linear and nonlinear degenerate points.

2.3.1 Decomposing the Tensor Field

First, recall from section 1.4.1 that real symmetric second order tensors can be understood as symmetric linear maps. For this section, we will assume a fixed orthonormal basis of \mathbb{R}^2 , so each symmetric linear map is represented by a 2×2 symmetric matrix M . Then, we can decompose M into an isotropic part γI_2 and a deviator part D :

$$M = \begin{pmatrix} \gamma + \alpha & \beta \\ \beta & \gamma - \alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D + \gamma I_2$$

Here, γ is the average of the two values on the diagonal of M and I_2 is the two-dimensional identity matrix. Because the trace of a matrix is equal to the sum of its eigenvalues, γ also coincides with the average of the two eigenvalues of M .

In the decomposition above, γI_2 is indeed isotropic because both eigenvalues of γI_2 are equal to γ . Scalar multiples of the identity matrix are the only isotropic real symmetric tensors (because isotropic tensors have all equal eigenvalues and symmetric tensors have all real eigenvalues with orthogonal eigenvectors). Recall from section 1.5.1 that in two-dimensional tensor fields, degenerate tensors are equivalent to isotropic tensors.

Next, consider some degenerate point of the tensor field F . We can rewrite F such that our chosen degenerate point lies at the origin $(0, 0)$. Then, for each $q \in \mathbb{R}^2$ we can write the associated tensor in F as the matrix:

$$M(q) = D(q) + \gamma(q)I_2 = \begin{pmatrix} \alpha(q) & \beta(q) \\ \beta(q) & -\alpha(q) \end{pmatrix} + \gamma(q)I_2 \quad (8)$$

Here, α , β , and γ are real-valued functions of q , while M and D are matrix-valued functions of q . Note that at degenerate points p , we have $M(p) = \lambda I_2$, so $\alpha(p) = \beta(p) = 0$. Because $(0, 0)$ is a degenerate point by construction, we have $\alpha((0, 0)) = \beta((0, 0)) = 0$. Finally, because adding scalar multiples of the identity matrix does not affect the eigenvectors of a matrix, we have that a vector v is an eigenvector at q if and only if v is an eigenvector of the deviator $D(q)$. The eigenvalues of $D(q)$ are $\pm \sqrt{\alpha(q)^2 + \beta(q)^2}$.

2.3.2 The Invariant δ

We now introduce the invariant δ , using the partial derivatives of the decomposition found in the previous section. Suppose p is a degenerate point and the tensor field F is decomposed around p as defined in section 2.3.1. As in section 2.3.1, the degenerate point p now lies at the origin. We define [6]:

$$\delta = \det \begin{pmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{pmatrix} \bigg|_{(x,y)=(0,0)} \quad (9)$$

In other words, δ is the determinant of the Jacobian matrix at the origin for the function $(x, y) \mapsto (\alpha((x, y)), \beta((x, y)))$. It can be shown that δ is invariant under any rotation of the tensor field's coordinate system [3].

The invariant δ is used to distinguish between two types of degenerate points: linear ($\delta \neq 0$) and nonlinear ($\delta = 0$) degenerate points. In Delmarcelle's original dissertation [3], Delmarcelle calls these points simple degenerate points and multiple degenerate points respectively. The following sections will study these two types of degenerate points.

2.4 Linear Degenerate Points

Linear degenerate points satisfy $\delta \neq 0$, where δ is the invariant introduced in the previous section. Using the notation from Equation 8, recall that the degenerate point of interest lies at the origin.

Then, because $\delta \neq 0$, by choosing a small enough neighborhood around the origin, the functions α and β can be approximated by the linear functions:³

$$\begin{aligned}\alpha((x, y)) &\approx \alpha_1 x + \alpha_2 y \\ \beta((x, y)) &\approx \beta_1 x + \beta_2 y\end{aligned}\tag{10}$$

Therefore, around the degenerate point (the origin), we can treat F as a linear tensor field, a tensor field where each component depends linearly on $q \in \mathbb{R}^2$. Note that a linear tensor field has only one degenerate point p where $\alpha(p) = \beta(p) = 0$. In our case, the unique degenerate point is at the origin.

2.4.1 Boundaries of Sectors: the Linear Case

In section 2.2, we defined hyperbolic, parabolic, and elliptic sectors around a degenerate point. In this section, we show that boundaries between sectors can be computed relatively easily in the linear case.

To begin, we show that the eigenvectors at a point q in a linear tensor field F depend only on the angle between q and the x -axis. In other words, the eigenvectors are independent of the distance between q and the origin. Suppose v is an eigenvector at point $q \in \mathbb{R}^2$. Using the notation from Equation 8:

$$\begin{aligned}D(q) \cdot v &= \lambda v \\ mD(q) \cdot v &= m\lambda v & \forall m \in \mathbb{R} \\ D(mq) \cdot v &= m\lambda v & (D \text{ is linear because its entries } \alpha \text{ and } \beta \text{ are linear})\end{aligned}$$

Thus, v is an eigenvector at all scalar multiples of q , so the eigenvectors at q depend only on the angle between q and the x -axis. We can then consider eigenvectors for each angle θ around the degenerate point rather than for each individual point q . Thus, we consider the eigenvectors for:

$$D_\theta = D((\cos \theta, \sin \theta)) = \begin{pmatrix} \alpha((\cos \theta, \sin \theta)) & \beta((\cos \theta, \sin \theta)) \\ \beta((\cos \theta, \sin \theta)) & -\alpha((\cos \theta, \sin \theta)) \end{pmatrix}$$

Then, we return to the topic of sector boundaries for linear degenerate points. It can be shown that boundaries between sectors of linear degenerate points lie on radial directions, angles θ where one of the eigenvectors also makes the angle θ with the x -axis (see Figure 3) [6]. Equivalently, radial directions are angles from which tensor field lines approach the degenerate point along straight lines. In order to find radial directions, we only need to find angles θ where $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ is an eigenvector of D_θ . Equivalently:⁴

$$\left\| \begin{pmatrix} \alpha((\cos \theta, \sin \theta)) & \beta((\cos \theta, \sin \theta)) \\ \beta((\cos \theta, \sin \theta)) & -\alpha((\cos \theta, \sin \theta)) \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \times \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right\| = 0$$

By substituting Equation 10, we eventually obtain:

$$\tan 2\theta = \frac{\beta_1 \cos \theta + \beta_2 \sin \theta}{\alpha_1 \cos \theta + \alpha_2 \sin \theta}$$

³We assume here that α and β are both analytic functions, i.e. they can be expressed as power series in some neighborhood around any point in their domain. Because $\delta \neq 0$, at least one value in each row and each column of the Jacobian matrix in Equation 9 must be nonzero. In other words, α and β both have at least one non-vanishing coefficient for a first degree term in their power series around the origin, and together these non-vanishing terms correspond to at least one first degree term for both x and y . Thus, we can choose a neighborhood around the origin small enough that terms of degree greater than one are negligible in comparison to the terms of degree one (e.g. x^2 and y^2 are negligible compared to x and y). Then, both α and β can be approximated using only their first degree terms. In Equation 10, we have $\alpha_1 = \frac{\partial \alpha}{\partial x}$, $\alpha_2 = \frac{\partial \alpha}{\partial y}$, $\beta_1 = \frac{\partial \beta}{\partial x}$, and $\beta_2 = \frac{\partial \beta}{\partial y}$, where all partial derivatives are evaluated at the origin.

⁴As in [6], the cross product of $v_1, v_2 \in \mathbb{R}^2$ is defined as the usual three-dimensional cross product when setting $z = 0$ for both vectors. Then, $\|v_1 \times v_2\| = \|v_1\| \|v_2\| \sin \theta$, where θ is the angle between v_1 and v_2 . Equivalently, $\|v_1 \times v_2\|$ is the area of the parallelogram defined by v_1 and v_2 . Assuming $v_1, v_2 \neq \vec{0}$, the magnitude of the cross product is equal to zero if and only if v_1 and v_2 point in the same (or opposite) direction (i.e. $v_1 = \lambda v_2$).

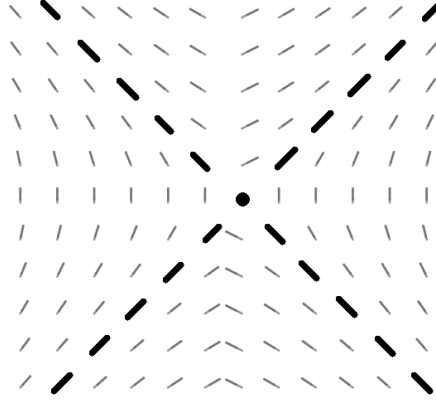


Figure 3: Line segments represent eigenvector directions at each point. Bolded line segments indicate radial directions around the degenerate point. In this figure, there are four radial directions.

By substituting $u = \tan \theta$, we obtain:

$$\beta_2 u^3 + (\beta_1 + 2\alpha_2)u^2 + (2\alpha_1 - \beta_2)u - \beta_1 = 0 \quad (11)$$

Thus, there are up to three real solutions $u = \tan \theta$. Because the tangent function has period π , each of these solutions corresponds to two radial directions: θ and $\theta + \pi$.

Finally, for each of these pairs of radial directions, we show that one angle is a radial direction in the major eigenvector field and the other angle is a radial direction in the minor eigenvector field. First, note that θ and $\theta + \pi$ lie on opposite sides of the unit circle, but the angle of the radial eigenvector is the same (because eigenvectors are bi-directional). Also note that if a point q makes angle θ with the x -axis, then $-q$ makes angle $\theta + \pi$ with the x -axis. Suppose v is a radial eigenvector of both q and $-q$. We have that:

$$\begin{aligned} D(q) \cdot v &= \lambda v \\ \iff -D(q) \cdot v &= -\lambda v \\ \iff D(-q) \cdot v &= -\lambda v \quad (D \text{ is linear because its entries } \alpha \text{ and } \beta \text{ are linear}) \end{aligned}$$

Recall from section 2.3.1 that every deviator matrix has one positive and one negative eigenvalue. From the equation above, at $-q$ (the angle $\theta + \pi$), the radial eigenvector v corresponds with the opposite sign eigenvalue from at q (the angle θ). Then, the isotropic part of the matrix adds a constant value to both eigenvalues (without changing the eigenvectors) and thus does not change which eigenvector is major or minor. Therefore, if the radial eigenvector v is a major eigenvector for θ , then it is a minor eigenvector for $\theta + \pi$, and vice versa. Because the eigenvector fields are smooth (except at the origin, the only degenerate point) and the two eigenvalues are only equal at the origin, all eigenvectors for the angle θ belong to one eigenvector field, and all eigenvectors for the angle $\theta + \pi$ belong to the other eigenvector field.

Therefore, the solutions to Equation 11 each correspond to exactly one radial direction in each eigenvector field (major and minor). Hence, every linear degenerate point has one, two, or three radial directions in each eigenvector field. Because boundaries between sectors lie on radial directions, linear degenerate points can have at most three sectors in any eigenvector field. As a last note, recall that sector boundaries that border a hyperbolic sector are called separatrices. By identifying sector boundaries of linear degenerate points, this section has also shown how to identify the separatrices of a linear degenerate point.

2.4.2 Types of Linear Degenerate Point

As it turns out, much more can be said about the sectors of a linear degenerate point, particularly using the invariant δ introduced in section 2.3.2. In his dissertation [3], Delmarcelle shows that if $\delta \neq 0$, then the tensor index $I_F(p) = \frac{1}{2}\text{sign}(\delta)$. Combining this with Equation 7 from section 2.2,

we obtain:

$$\begin{aligned} \frac{1}{2}\text{sign}(\delta) &= 1 + \frac{n_e - n_h}{2} \\ n_h &= 2 - \text{sign}(\delta) + n_e \end{aligned} \quad (12)$$

Here, n_h is the number of hyperbolic sectors, and n_e is the number of elliptic sectors. Delmarcelle also shows that when moving along a closed curve enclosing a single linear degenerate point, eigenvectors rotate in a constant direction. Using these facts, we can characterize the two types of linear degenerate point: wedge points ($\delta > 0$, $I_F(p) = \frac{1}{2}$) and trisector points ($\delta < 0$, $I_F(p) = -\frac{1}{2}$). Note that because δ remains the same for both the major and minor eigenvector fields, a wedge point is a wedge point in both eigenvector fields, and similarly for a trisector point.

2.4.3 Types of Linear Degenerate Point: Wedge Points

First, we consider some wedge point p_w , a linear degenerate point where $\delta > 0$. We claim that p_w has only one hyperbolic sector and no elliptic sectors. From Equation 12, we have $n_h = 1 + n_e$, so it suffices to show that $n_e = 0$. Suppose to the contrary that $n_e \geq 1$. Then, $n_h \geq 2$. From section 2.2, eigenvectors rotate counterclockwise in the elliptic sector. Because the direction of rotation is constant around a linear degenerate point, and because eigenvectors in hyperbolic sectors rotate clockwise when the angle of the sector is less than π , each hyperbolic sector must have angle greater than or equal to π . This is a contradiction because there are at least two hyperbolic sectors, and the elliptic sector has angle greater than zero. Therefore, p_w has no elliptic sectors and one hyperbolic sector.

Because we showed in section 2.4.1 that linear degenerate points have at most three sectors, we have two possibilities:

- The degenerate point p_w has one hyperbolic sector and no parabolic sectors. Then, p_w has one separatrix as shown in Figure 4a.
- The degenerate point p_w has one hyperbolic sector and one or two parabolic sectors. Because we combine adjacent parabolic sectors, p_w has one hyperbolic sector and one parabolic sector. Because eigenvectors rotate counterclockwise in parabolic sectors, the hyperbolic sector must have angle greater than or equal to π . In this case, p_w has two separatrices as shown in Figure 4b.

2.4.4 Types of Linear Degenerate Point: Trisector Points

Then, we consider some trisector point p_t , a linear degenerate point where $\delta < 0$. From Equation 12, we have $n_h = 3 + n_e$. Because we showed in section 2.4.1 that linear degenerate points have at most three sectors, p_t has three hyperbolic sectors and no other sectors. Recall from section 2.2 that a hyperbolic sector with angle θ rotates eigenvectors counterclockwise by $\theta - \pi$. Because eigenvectors rotate in a constant direction around a linear degenerate point, and because all three sectors cannot each have angle greater than or equal to π , each hyperbolic sector must have angle less than or equal to π . An example of a trisector point is shown in Figure 4c. A trisector point has three separatrices.

2.5 Nonlinear Degenerate Points

Nonlinear degenerate points satisfy $\delta = 0$, where δ is the invariant introduced in section 2.3.2. Because the tensor field around a nonlinear degenerate point cannot be approximated as a linear tensor field, it is difficult to characterize all nonlinear degenerate points. However, it can be shown that a nonlinear degenerate point p is equivalent in the far field to a combination of linear degenerate points whose tensor indices sum to $I_F(p)$. Formally, given an open set $L \subset \mathbb{R}^2$ containing a single nonlinear degenerate point, the tensors corresponding to points in L can be replaced such that L contains only simple linear degenerate points and tensors on the boundary and exterior of L remain unchanged [3].

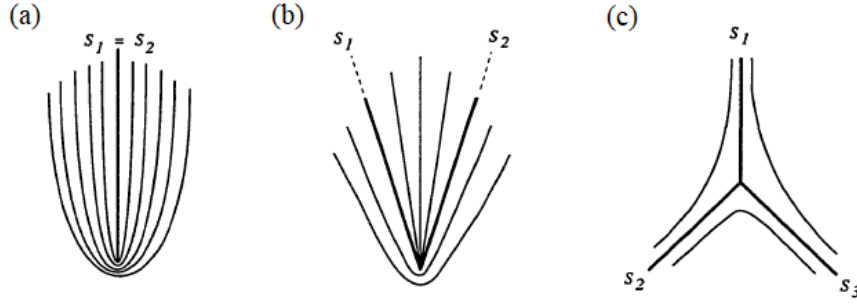


Figure 4: Examples of a wedge point with one separatrix (a), a wedge point with two separatrices (b), and a trisector point (c). Figure taken from [3].

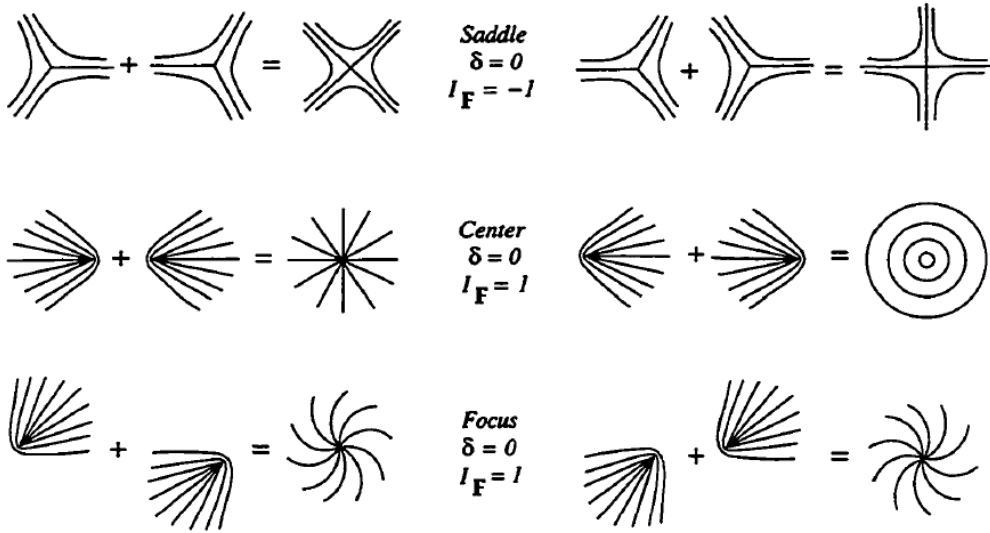


Figure 5: Examples of merging degenerate points. The left and right columns depict the major and minor eigenvector fields. The top row depicts merging trisector points. The bottom two rows depict merging wedge points, resulting in a center or focus point. Figure taken from [3].

Because nonlinear degenerate points can be replaced with multiple linear degenerate points (and vice versa) at arbitrarily small scales, it follows that tensor fields that depend on some continuous parameter (such as time) often exhibit merging of linear degenerate points and decomposing of nonlinear degenerate points. Examples of merging degenerate points are shown in Figure 5. Note that the tensor index of a path surrounding the region of interest is unchanged by merges and decompositions.

2.5.1 Centers and Foci

Centers and foci are mentioned specifically here because they have no hyperbolic, parabolic, or elliptic sectors. At a center point, there are no radial directions in the given eigenvector field, but in the other eigenvector field, all angles are radial directions (a star point). At a focus point, tensor field lines rotate around the degenerate point without ever reaching it, creating a swirl pattern. Because the major and minor eigenvector fields are orthogonal, the two eigenvector fields at a focus point correspond to two foci with opposite swirl direction. Both centers and foci have tensor index one. A center and focus point are shown in Figure 5.

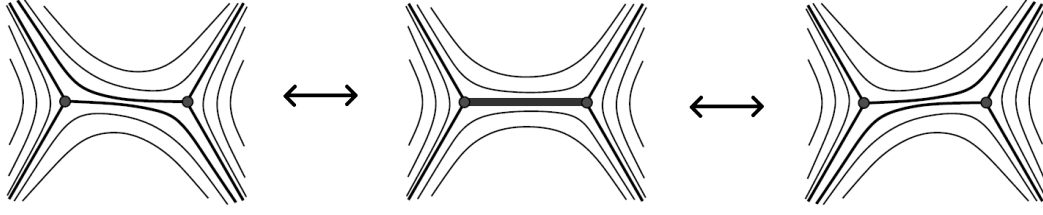


Figure 6: Examples of unstable separatrices. Separatrices in close proximity to one another can merge into one separatrix (connecting the two degenerate points, as shown in the middle case) and can dramatically alter tensor field lines in the region. Figure taken from [6].

2.6 Structural Stability and Bifurcations

As explained in section 2.5, tensor fields that depend on some continuous parameter often exhibit merging and decomposing of degenerate points. In this section, we assume that our tensor field depends on the continuous parameter t . This section considers how a tensor field's topology changes with t . In general, we will not formally prove the results in this section, relying instead on a qualitative study of eigenvector fields, as in [6].

2.6.1 Locally Stable and Unstable Structures

A degenerate point or separatrix is called locally stable if and only if it persists through small perturbations in t . A nonlinear degenerate point p is not locally stable because it can be decomposed into a set of linear degenerate points by changing an arbitrarily small region around p (intuitively, then the degenerate point can decompose over an arbitrarily small perturbation in t).

However, individual linear degenerate points are locally stable. Linear degenerate points cannot be decomposed into degenerate points with lower tensor index because tensor index values are always half-integers. Then, both wedges and trisectors are locally stable, corresponding to tensor indices $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. An existing wedge or trisector point can only be altered by merging with another degenerate point.

Finally, consider a separatrix connecting two degenerate points p_0 and p_1 . Recall that a separatrix bounds a hyperbolic sector. Thus, the sector on one side of the separatrix must be hyperbolic. If the sectors on both sides of the separatrix are hyperbolic, then the separatrix is locally unstable. If an arbitrarily small angular perturbation is made at any point along the separatrix, then the connection between p_0 and p_1 is broken, as shown in Figure 6.

2.6.2 Bifurcations

Merges and decompositions of degenerate points are examples of bifurcations, changes in the tensor field's degenerate points or separatrices given a change in t . Bifurcations can be local or global, either inducing changes restricted to a local region (where "local" depends on the area of interest) or inducing changes to larger regions of the tensor field domain.

In general, because degenerate point merging and decomposition can occur in an arbitrarily small region, degenerate point bifurcations are local. In particular, two important types of local degenerate point bifurcations have yet to be mentioned: pairwise creations and pairwise annihilations. In a pairwise creation event, a region containing no degenerate points transitions into a region containing one wedge (index $I_F(p) = \frac{1}{2}$) and one trisector (index $I_F(p) = -\frac{1}{2}$) point. In a pairwise annihilation event, a region containing one wedge and one trisector point transitions into a region containing no degenerate points. Both of these transitions are topologically plausible because they preserve the tensor index zero around the region of interest. A pairwise creation and annihilation event is depicted in Figure 7.

In contrast to degenerate point bifurcations, most bifurcations involving separatrices are global. Changes in separatrices affect essentially all tensor field lines in a region. As shown in Figure 6, separatrices that pass close to one another may lead to global bifurcations.

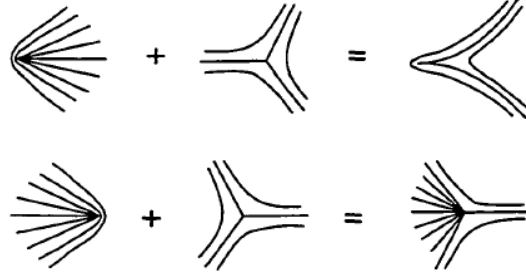


Figure 7: Examples of pairwise creation or annihilation events, which each involve one wedge and one trisector point. The resulting nonlinear degenerate points on the right have tensor index zero, and can thus be replaced by a region containing no degenerate points. Figure taken from [3].

2.7 Computing the Tensor Field Topology

In practice, it is difficult to precisely compute the degenerate points and separatrices of an arbitrary tensor field. For instance, when computing degenerate points, it may be tempting to simply compute major and minor eigenvalues at all points on some grid, interpolating to find points where the eigenvalues are equal. This is ineffective because the major eigenvalue is always greater than the minor eigenvalue, so the interpolated eigenvalues will never be equal. Alternatively, instead of sorting into major and minor eigenvalues, it could be tempting to separate eigenvalues into two groups by assuming that each eigenvalue group should be differentiable in \mathbb{R}^2 ; however, the underlying differentiability assumption is not met because eigenvalue fields may be non-differentiable at degenerate points.

Furthermore, it is difficult to compute tensor field lines in an arbitrary tensor field. In vector fields, streamlines are often computed in practice by repeatedly taking steps in the direction of the vector at the current point. In vector fields, step size is equal to some small base step size parameter multiplied by the current vector's magnitude. However, eigenvectors do not have magnitudes, and they always point in two opposing directions. Choosing the correct direction to step in an eigenvector field is often solved by choosing the direction that minimizes the angle with the previous step's direction. Choosing an effective step size when integrating tensor field lines remains an open problem, because eigenvectors can quickly change direction in the vicinity of a degenerate point [6]. For this paper, we assume a small step size when integrating tensor field lines in practice.

2.7.1 Techniques for Computing the Tensor Field Topology

This section outlines a method proposed by Delmarcelle [3] for computing a tensor field's degenerate points and separatrices. To find degenerate points, the tensor field is split into cells; tensor field values are computed on the four corners of cells, and bilinear interpolation is used to find tensor field values within the cell.⁵ This method treats the tensor field within each cell as a linear tensor field, where the unique possible degenerate point can be computed using linear and quadratic equations. Linear tensor fields were described in detail in section 2.4.

Because the interpolation method was linear, all degenerate points found this way are linear. Then, the determinant δ (see section 2.3.2) is used to identify wedge and trisector points (see section 2.4.2). The method from section 2.4.1 is used to identify radial directions and separatrices. Finally, tensor field lines are integrated along the desired separatrix directions. The resulting topology (degenerate points and separatrices) is used as the base topology for the simplification methods introduced in the next section.

⁵Bilinear interpolation is an extension of linear interpolation used for a function f of two variables. Essentially, linear interpolation is used along the top and bottom (y_0 and y_1) boundaries to obtain two interpolated values $\hat{f}(x, y_0)$ and $\hat{f}(x, y_1)$ at the desired x . Then, linear interpolation is used again to find the interpolated value $\hat{f}(x, y)$ at the desired y .

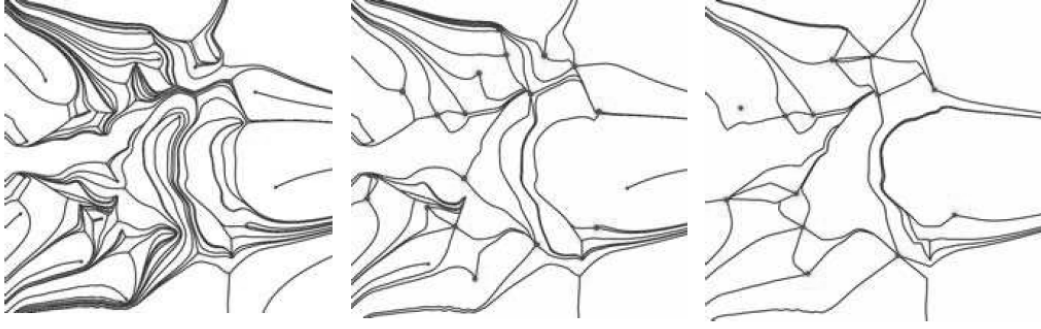


Figure 8: An example of simplifying a tensor field topology using topology scaling. Figure taken from [6].

2.8 Topology Simplification

When computing the topology of a tensor field, it is often desirable to extract only degenerate points and separatrices that represent “important” features of the data. For instance, in datasets where there are multiple structures at various scales, there are often many degenerate points and separatrices, not all of which are relevant for any given purpose. In fact, some degenerate points may be artifacts of bilinear interpolation or approximation in the numerical simulation. The following sections will consider methods of simplifying the topology of a tensor field.

2.8.1 Topology Scaling

Topology scaling is a topology simplification method that assumes that complexities in the topology are inherent to the dataset. In other words, the degenerate points and separatrices at small scales are relevant to the dataset, but irrelevant at the desired scale.

In topology scaling, degenerate points are clustered into regions, where degenerate points in any given region remain close enough to satisfy some proximity criterion. This initial step can use a variety of clustering algorithms, which we will not describe here. Next, each region is transformed such that it contains only a single degenerate point, leaving the tensor field values on the region boundaries unchanged. Such a transformation is guaranteed to exist, as noted in section 2.5. The resulting set of degenerate points is taken to be the set of “relevant” degenerate points of the tensor field. Finally, radial directions are computed for each degenerate point, allowing sector boundaries and separatrices to be computed as usual.

As a note, this method of topology simplification is highly dependent on the choice of proximity criterion. This dependence makes intuitive sense; the set of “relevant” degenerate points should be dependent on the desired granularity of the simplified topology. An example of topology scaling is shown in Figure 8.

2.8.2 Continuous Topology Simplification

Continuous topology simplification is a simplification method that assumes that complexities in the topology are insignificant to the actual tensor field topology. For instance, the complexities may be caused by numerical approximations or the interpolation scheme. At a high level, continuous topology simplification repeatedly combines wedge and trisector points using pairwise annihilation events (see section 2.6.2).

Recall from section 2.7.1 that due to bilinear interpolation, all degenerate points found in the original computed topology are linear (i.e. wedges or trisectors). In continuous topology simplification, all degenerate points are assigned to pairs, each containing one wedge and one trisector point. Pairs are assigned scalar values based on some desired criterion (e.g. low values could indicate points that are close together). Then, pairs are sorted by this scalar value and processed sequentially. During processing, a region is found that contains only the wedge and trisector point in the pair. Because the overall tensor index of this region is zero, the region

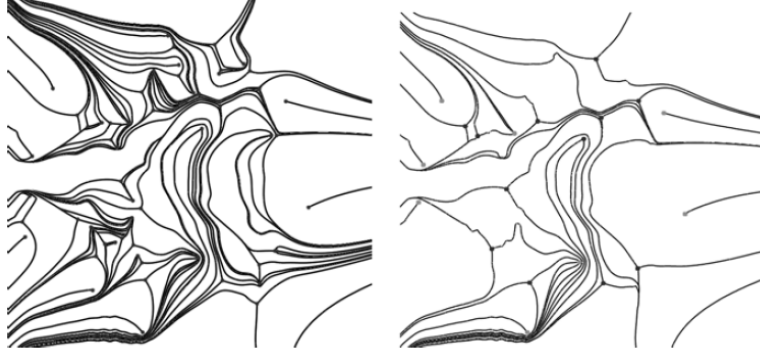


Figure 9: An example of simplifying a tensor field topology using continuous topology simplification. Figure taken from [6].

can be transformed such that it contains no degenerate points and the tensor field values on the boundary remain unchanged.

In this way, continuous topology simplification removes wedge and trisector points that do not surpass some relevance threshold defined by the user. An example of continuous topology simplification is shown in Figure 9.

2.9 Topology Tracking

The previous sections only outline methods of computing the topology of a static (not parameter-dependent) tensor field. When considering parameter-dependent tensor fields such as in section 2.6, it is important to understand the changes within a tensor field with respect to a parameter t . Visualizing the continuous change of a tensor field topology with respect to t is called topology tracking.

One method of topology tracking is described in [6]. The parameter t is considered as an additional dimension to the tensor field, resulting in a space-time grid of t (time) and the tensor space (\mathbb{R}^2). Initially, it is assumed that the tensor field topologies for all values of t are computed using the same bilinear interpolation scheme, using triangular cells. Each triangular cell is tracked over time.

As explained in section 2.7.1, each cell at any given time can contain at most one degenerate point, and this degenerate point must be linear. Because linear degenerate points are stable (see section 2.6.1), a linear degenerate point on the interior of a cell cannot disappear over time, assuming the tensor field changes smoothly with respect to t . Thus, each degenerate point can be tracked over time on the interior of its cell.

However, degenerate points exhibit more interesting behavior when they reach the boundary of a cell. In simple cases, the degenerate point may just be moving from one cell to another. Then, the degenerate point is tracked within the new cell. Alternatively, the point may be involved in a pairwise annihilation with a degenerate point from a neighboring cell. Similarly, pairwise creation events can occur on the boundaries of cells, resulting in two degenerate points, one in each of two neighboring cells. Pairwise creation and annihilation events must occur on the boundaries of cells because both events involve more than one degenerate point (and each cell contains at most one degenerate point). This constraint simplifies the detection of pairwise creation and annihilation events, allowing all degenerate points and bifurcations to be tracked over time. Finally, as usual, separatrices are integrated from each degenerate point at each value of t . An example of the resulting topology evolution is shown in Figure 10.

3 Topology of Three-Dimensional Tensor Fields

The concepts introduced in section 2 are largely specific to two-dimensional tensor fields. This section will consider degenerate features of three-dimensional tensor fields (i.e. $V = \mathbb{R}^3$), focusing

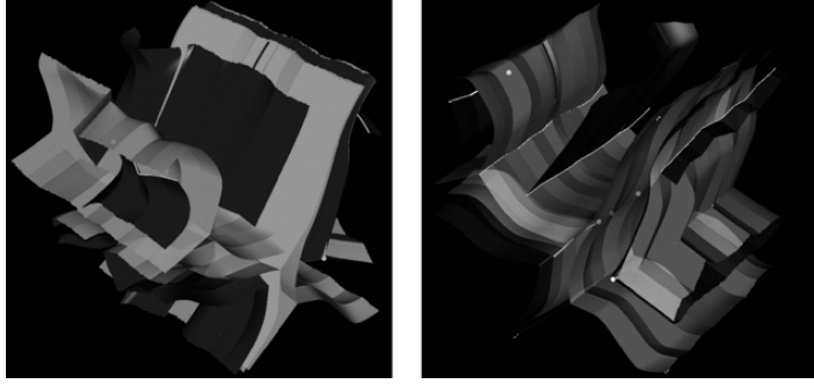


Figure 10: An example of applying topology tracking to visualize the evolution of a two-dimensional tensor field topology. Separatrices appear as surfaces, lines that change with respect to the dimension t . Figure taken from [6].

on methods of identifying these degenerate features. In particular, degenerate tensors (at least two equal eigenvalues) are equivalent to isotropic tensors (all equal eigenvalues) in 2D tensor fields; this is clearly not the case in 3D tensor fields, where all tensors have three real eigenvalues (assuming real symmetric tensors as before). The three eigenvector fields are called the minor, medium, and major eigenvector fields, corresponding to eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$. A linear degenerate point occurs when the minor and medium eigenvalues are equal; a planar degenerate point occurs when the medium and major eigenvalues are equal. A triple degenerate point occurs when all three eigenvalues are equal.

In section 3.1, we will justify an assertion that degenerate features in 3D form lines instead of isolated points. In section 3.2, we will outline a method for computing degenerate points using implicit functions, and in section 3.3, we describe a method for computing degenerate points using a geometric interpretation of degenerate tensors. Finally, we connect degenerate points into degenerate feature lines in section 3.4.

3.1 Dimensionality of Degenerate Spaces

In the two-dimensional tensor fields studied in section 2, degenerate spaces typically took the form of isolated degenerate points (zero-dimensional). In this section, we will provide a high level justification for this result, applying the same reasoning to conclude that degenerate spaces in three-dimensional tensor fields are often one-dimensional.

First, the space \mathcal{S}_2 of real symmetric two-dimensional tensors has dimension three because it takes three independent values to fully specify a real symmetric 2×2 matrix. Next, a real symmetric two-dimensional tensor field F_2 typically forms a subspace $\mathcal{F}_2 \subseteq \mathcal{S}_2$ with dimension two because there are two dimensions in the tensor space \mathbb{R}^2 , and each point in the tensor space is assigned exactly one tensor. Finally, the set of degenerate tensors forms a subspace $\mathcal{D}_2 \subseteq \mathcal{S}_2$ with dimension one because these tensors must have matrix representations λI_2 (see section 2.3.1).

Then, the codimension of \mathcal{F}_2 in \mathcal{S}_2 is one, and the codimension of \mathcal{D}_2 in \mathcal{S}_2 is two. Note that the set $\mathcal{F}_2 \cap \mathcal{D}_2$ represents the set of degenerate points of F_2 . In typical scenarios [7], we have $\text{codim}(\mathcal{F}_2 \cap \mathcal{D}_2) = \text{codim}(\mathcal{F}_2) + \text{codim}(\mathcal{D}_2) = 3$. Although not a formal justification, we can refer to the analogy in which a two-dimensional curve (codimension one, e.g. a plane) usually intersects a one-dimensional curve (codimension two, e.g. a line) at isolated points (codimension three) in three-dimensional space. Then, $\mathcal{F}_2 \cap \mathcal{D}_2$ typically has dimension zero, and the degenerate points of F_2 typically form isolated points.

We then apply the same reasoning to three-dimensional tensor fields. The space \mathcal{S}_3 of real symmetric three-dimensional tensors has dimension six because it takes six independent values to fully specify a real symmetric 3×3 matrix. A real symmetric three-dimensional tensor field F_3 typically forms a subspace $\mathcal{F}_3 \subseteq \mathcal{S}_3$ with dimension three because there are three dimensions in the tensor space \mathbb{R}^3 . The set of degenerate tensors forms a subspace $\mathcal{D}_3 \subseteq \mathcal{S}_3$ with dimension

four; this dimensionality will be addressed in section 3.3.1. Then, in typical scenarios, we have $\text{codim}(\mathcal{F}_3 \cap \mathcal{D}_3) = \text{codim}(\mathcal{F}_3) + \text{codim}(\mathcal{D}_3) = 3 + 2 = 5$. Then, $\mathcal{F}_3 \cap \mathcal{D}_3$ typically has dimension one, so the degenerate points of F_3 typically form lines.

Indeed, stable degenerate features of 3D tensor fields have often been found to form lines; degenerate features forming isolated points, surfaces, or subvolumes are usually unstable, disappearing under small perturbations [7]. Thus, the following sections will focus on finding degenerate feature lines in 3D tensor fields. At a high level, this is often accomplished by first finding degenerate points, then connecting these points to identify degenerate feature lines.

3.2 Finding Degenerate Points Using Implicit Functions

The first method to finding degenerate points relies on implicit functions, functions which equal zero if and only if the input tensor is degenerate. For instance, suppose a tensor T has matrix representation M with eigenvalues λ_1 , λ_2 , and λ_3 . The discriminant $D_3(T)$ is defined as:

$$D_3(T) = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2 \quad (13)$$

Note that $D_3(T) = 0$ if and only if T has at least two equal eigenvalues (i.e. T is degenerate).

Using the discriminant equation above to identify degenerate tensors would require all eigenvalues to be computed for each tensor, which is often computationally expensive. Instead, $D_3(T)$ can be computed directly from the tensor's matrix entries, using a degree six polynomial. For the entire written out polynomial, see [7]. Then, roots of this polynomial $D_3(T)$ correspond to degenerate tensors. Unfortunately, in practice it is often difficult to find roots of this polynomial. Because $D_3(T)$ is always non-negative, its roots occur at its minima; thus, common root-finding algorithms that detect changes in sign are ineffective. Furthermore, minimization algorithms such as gradient descent are often ineffective because gradients have been found to be unstable except when very close to the degenerate tensor [7].

Thus, it is desirable to further simplify the implicit function $D_3(T)$. It can be shown that $D_3(T)$ can be written as the sum of the squares of seven cubic polynomials (again in terms of the tensor's matrix entries). As before, refer to [7] to see the written out polynomials. Because each squared polynomial is non-negative, $D_3(T)$ is equal to zero if and only if all seven cubic polynomials are equal to zero. Finding roots of these cubic polynomials is more efficient than the previous discriminant-based methods because we only need to find roots of cubic polynomials (instead of roots of a degree six polynomial, and we still do not have to compute eigenvalues). In particular, because the cubic polynomials can be both positive and negative, it is much simpler to test for the existence of degenerate points in any given region.

In practice, the tensor space (\mathbb{R}^3) is separated into hexahedral (six-sided) cells. Roots of the seven cubic polynomials are found on each face of each cell. This can be done using variations on standard root-finding algorithms that attempt to satisfy all seven constraints simultaneously. Note that these algorithms must be altered slightly because we are attempting to find zeroes of seven polynomials, but there are only two independent directions on any given cell face. For instance, the iterative method proposed in [7] minimizes the squares of all seven polynomials, while only moving throughout the cell face. The details of this root-finding algorithm will not be described here, but the algorithm is effective at finding degenerate points with high degrees of precision and low false negative rates [7].

3.3 Finding Degenerate Points Using the Geometric Approach

None of the methods described in section 3.2 result in a system with the same number of equations as unknowns, the types of systems that are ideal for most optimization algorithms. For instance, using the discriminant alone minimizes one equation (the discriminant) over two unknowns (the two independent directions on any given cell face). Using the seven cubic polynomials finds roots of seven equations over the same two unknowns. In this section, we will describe a method that finds degenerate points using a system of six equations with six unknowns.

3.3.1 Geometric Representation of Degenerate Tensors

First, we describe a geometric interpretation of degenerate tensors, showing that degenerate tensors are specified by four independent parameters. We will show that a tensor is degenerate if and only if its matrix representation M can be written in the form $M = sI_3 \pm VV^T$, where I_3 is the three-dimensional identity matrix, $s \in \mathbb{R}$, and $V \in \mathbb{R}^3$.

(\Leftarrow) Suppose that M can be written $M = sI_3 \pm VV^T$. Note that VV^T represents a linear transformation that projects the input vector onto V , then scales the resulting vector by $\|V\|^2$.⁶ Thus, $\pm VV^T$ has two zero eigenvalues, and the remaining eigenvalue is equal to $\pm\|V\|^2$. Adding sI_3 simply adds s to every eigenvalue of $\pm VV^T$, so M has two eigenvalues equal to s and one eigenvalue equal to $s \pm \|V\|^2$. Because M has at least two equal eigenvalues, the tensor represented by M is degenerate.

(\Rightarrow) Suppose that a degenerate tensor has matrix representation M . If M has three equal eigenvalues, then $M = sI_3$ and we are done. Then we assume that M has exactly two equal eigenvalues, which we will denote s . Suppose we subtract sI_3 from M . Because this operation subtracts s from every eigenvalue of M , the remaining matrix $M - sI_3$ has two zero eigenvalues and one nonzero real eigenvalue. In other words, $M - sI_3$ projects all vectors onto a single line, so we say $M - sI_3$ represents a linear tensor. It can be shown that all real symmetric linear tensors can be represented as $\pm VV^T$ for some vector V . Then, we can write $M = sI_3 \pm VV^T$.

This geometric representation of degenerate tensors allows us to identify all three eigenvalues of a degenerate tensor. If $M = sI_3 + VV^T$, then the eigenvalues are s , s , and $s + \|V\|^2$. If $M = sI_3 - VV^T$, then the eigenvalues are s , s , and $s - \|V\|^2$. This also allows us to easily differentiate between linear degenerate points (minor and medium eigenvalues equal) and planar degenerate points (medium and major eigenvalues equal). Geometrically, M scales vectors equally in two dimensions, but differently in one orthogonal dimension. If this unequal dimension scales more than the other two dimensions, then we have a linear degenerate point. If the unequal dimension scales less than the other two dimensions, then we have a planar degenerate point.

3.3.2 Finding Degenerate Points Using the Geometric Representation

Similar to section 3.2, the tensor space (\mathbb{R}^3) is separated into hexahedral cells, and degenerate points are found on each face of each cell. Finding degenerate points is equivalent to finding points where $T(x, y)$ can be written $sI_3 \pm VV^T$. Thus, we are finding solutions of the form x, y, s, V , which consists of six independent unknowns (because $V \in \mathbb{R}^3$ and all other parameters are in \mathbb{R}). Because three-dimensional real symmetric tensors have six independent components (six equations), we can create a system with six equations and six unknowns. These types of systems with equal numbers of equations and unknowns typically have stable and isolated solutions.

Then, we can use any root-finding algorithm to compute degenerate points on a given cell face. For instance, we can use an iterative root-finding algorithm, using the center of the cell face as an initial guess and using some heuristic for the initial guesses s_0, V_0 .⁷ These iterative root-finding algorithms have been successful at finding degenerate points in practice, although they are less stable than the implicit function methods (see section 3.2) when near a triple degenerate point [7].

3.4 Computing Degenerate Feature Lines

As described in section 3.1, degenerate features in three-dimensional tensor fields typically form lines. Thus, once we have identified degenerate points, these degenerate points must be connected to form lines.

⁶Applying VV^T to a vector $A \in \mathbb{R}^3$ results in the output vector $VV^T A$. Because matrix multiplication is associative and $V^T A \in \mathbb{R}$, the output vector is a scalar multiple of V . Thus, VV^T has two zero eigenvalues (corresponding to the plane normal to V) and one nonzero real eigenvalue (corresponding to the direction of V). Because the trace of a matrix is equal to the sum of its eigenvalues, the remaining real eigenvalue is equal to $\|V\|^2$. Finally, by writing out VV^T , we can see that VV^T is symmetric.

⁷Reference [7] uses the Newton-Raphson method to find planar degenerate points, using the initial guesses $s_0 = \frac{\lambda_2 + \lambda_3}{2}$ and $V_0 = e_1 \sqrt{s_0 - \lambda_1}$, where $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues at the cell face's center (x_0, y_0) . The eigenvector e_1 corresponds with the eigenvalue λ_1 .

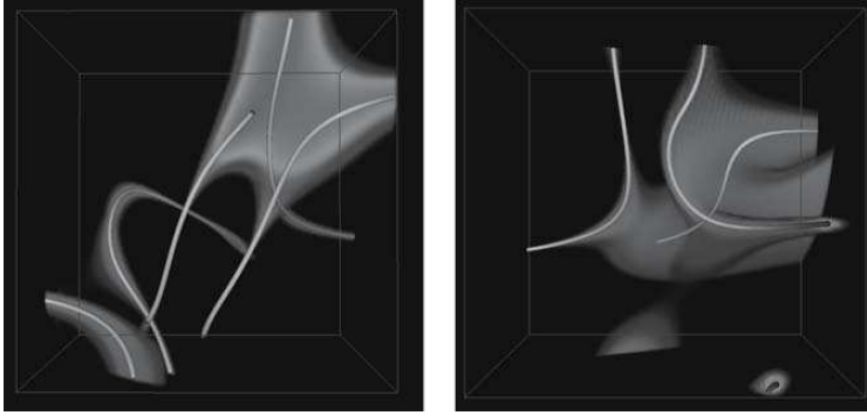


Figure 11: Examples of using the geometric representation to find degenerate features in randomly-generated three-dimensional tensor fields. Lines represent computed degenerate feature lines, and shading indicates areas close to these degenerate features (areas where the discriminant $D_3(T)$ is close to zero; see Equation 13). Figure taken from [7].

Using the method in [7], only cells containing exactly two degenerate points are initially considered. The two degenerate points in each such cell are connected by a straight line segment. These cells are marked as processed. Then, in each subsequent iteration, we consider a cell that has at least one degenerate point; one of these degenerate points is connected to a degenerate point in a nearby processed cell. If the current cell has more than one degenerate point, then the degenerate point is chosen such that it minimizes the angle between the new line segment and the processed cell's existing line segment. This process results in degenerate feature lines throughout the tensor space, as illustrated in Figure 11.

Of course, while this method is successful at identifying degenerate feature lines, there are several drawbacks to the method. Because this method is designed to identify degenerate feature lines specifically, when degenerate features form surfaces or subvolumes, the method produces unexpected degenerate feature lines around these surfaces and subvolumes. Additionally, this method does not compute structures connecting degenerate feature lines, analogous to the separatrices in section 2. These separating structures would likely form surfaces extending from degenerate feature lines, and they can be computed by sampling tensor field lines starting at points along each degenerate feature line.⁸

4 Conclusion

We have now outlined the topology of real symmetric second order tensor fields in \mathbb{R}^2 and \mathbb{R}^3 . Along with defining these topologies, we have characterized types of degenerate points in two-dimensional tensor fields, providing methods for computing these degenerate points and their associated separatrices. Additionally, we have described multiple ways of computing degenerate points in three-dimensional tensor fields, connecting these points to identify degenerate feature lines.

Before concluding this paper, we provide a brief description of tensor field applications to diffusion tensor imaging in neuroscience [1], hoping to justify further study of tensor field topology. In the brain, the Brownian motion of a water molecule can be modeled as a 3D Gaussian distribution centered at the origin. This 3D Gaussian distribution is defined entirely by its 3×3 covariance matrix D , and a diffusion MRI scan can approximately compute D for points through-

⁸Specifically, the tensor at some degenerate point along a degenerate feature line can be projected onto the plane corresponding to the two equal eigenvalues. Then, we have a two-dimensional degenerate tensor, similar to those studied in section 2, so separatrices can be computed extending from this point. This process can be repeated for degenerate points along the entire degenerate feature line, allowing us to compute separating structures in three-dimensional space.

out the brain. Then, D is called the diffusion tensor at a given point. Note that D is a real symmetric second-order tensor, so the set of diffusion tensors forms a real symmetric second-order tensor field in \mathbb{R}^3 , like those studied in section 3.

The eigenvectors and eigenvalues of the diffusion tensors in the brain provide useful information about brain structures, because these eigenvectors and eigenvalues correspond to principle directions of variance in the 3D Gaussian distribution. Directions of higher variance are directions in which the water molecule is more likely to move. As an example, gray matter in the brain generally contains more barriers to water movement than ventricles containing cerebro-spinal fluid. Thus, areas containing gray matter tend to have lower eigenvalues compared to ventricles, and these areas can be identified using diffusion MRI scans.

Furthermore, in white matter areas of the brain, axon fibers tend to restrict movement perpendicular to the cell wall, promoting motion in the direction of the axon fiber. Thus, the diffusion tensor at a point in an axon fiber has a higher eigenvalue in the direction of the fiber, with approximately equal (but lower) eigenvalues in the other two orthogonal directions. Notably, this corresponds to a degenerate point with exactly two equal eigenvalues, the precise type of point studied in section 3. Thus, computing the topology of a diffusion tensor field allows axon fibers to be identified in diffusion MRI scans.

Of course, tensor fields have applications far outside of neuroscience, in fields ranging from theoretical physics to computer graphics; a tensor can essentially represent any multilinear system over a vector space. Studying the topology of tensor fields has valuable applications to a variety of real world problems, and we hope that this paper has provided a useful introduction to the topology of second order tensor fields.

References

- [1] Alexander, D. (2006). An introduction to computational diffusion MRI: The diffusion tensor and beyond. In J. Weickert & H. Hagen (Eds.), *Visualization and processing of tensor fields* (pp. 83-106). Springer.
- [2] Chern, I. (2013). *Hilbert spaces*. Department of Mathematics, National Taiwan University. <http://ocw.nctu.edu.tw/course/ana021/AAchap4.pdf>
- [3] Delmarcelle, T. (1994). *The visualization of second-order tensor fields* [Doctoral dissertation, Stanford University]. ProQuest Dissertations Publishing.
- [4] Hagen, H., & Garth, C. (2006). An introduction to tensors. In J. Weickert & H. Hagen (Eds.), *Visualization and processing of tensor fields* (pp. 3-13). Springer.
- [5] Lay, D., Lay, S., & McDonald, J. (2016). *Linear algebra and its applications* (5th edition). Pearson.
- [6] Tricoche, X., Zheng, Z., & Pang, A. (2006). Visualizing the topology of symmetric, second-order, time-varying two-dimensional tensor fields. In J. Weickert & H. Hagen (Eds.), *Visualization and processing of tensor fields* (pp. 225-240). Springer.
- [7] Zheng, Z., Tricoche, X., & Pang, A. (2006). Degenerate 3D tensors. In J. Weickert & H. Hagen (Eds.), *Visualization and processing of tensor fields* (pp. 241-256). Springer.